6. *Finite Fields*

1. Identify the group $\mathbb{F}_4{}^{+}$.
2. Write out the addition and multiplication tables for $\mathbb{F}_4$ and for $\mathbb{Z}/(4)$, and compare them.
3. Find a thirteenth root of 3 in the field $\mathbb{F}_{13}$.
4. Determine the irreducible polynomial over $\mathbb{F}_2$ for each of the elements (6.12) of $\mathbb{F}_8$.
5. Determine the number of irreducible polynomials of degree 3 over the field $\mathbb{F}_3$.
6. **(a)** Verify that (6.9, 6.10, 6.13) are irreducible factorizations over $\mathbb{F}_2$.
   (b) Verify that (6.11, 6.13) are irreducible factorizations over $\mathbb{Z}$.
7. Factor $x^9 - x$ and $x^{27} - x$ in $\mathbb{F}_3$. Prove that your factorizations are irreducible.
8. Factor the polynomial $x^{16} - x$ in the fields **(a)** $\mathbb{F}_4$ and **(b)** $\mathbb{F}_8$.
9. Determine all polynomials $f(x)$ in $\mathbb{F}_q[x]$ such that $f(\alpha) = 0$ for all $\alpha \in \mathbb{F}_q$.
10. Let K be a finite field. Prove that the product of the nonzero elements of K is -1.
11. Prove that every element of $\mathbb{F}_p$ has exactly one pth root.
12. Complete the proof of Proposition (6.19) by showing that the difference $\alpha - \beta$ of two roots of $x^q - x$ is a root of the same polynomial.
13. Let p be a prime. Describe the integers n such that there exist a finite field K of order n and an element $\alpha \in K^\times$ whose order in $K^\times$ is p.
14. Work this problem without appealing to Theorem (6.4).
    (a) Let $F = \mathbb{F}_p$. Determine the number of monic irreducible polynomials of degree 2 in $F[x]$.
    (b) Let $f(x)$ be one of the polynomials described in (a). Prove that $K = F[x]/(f)$ is a field containing p^2 elements and that the elements of K have the form $a + b\alpha$, where $a, b \in F$ and α is a root of f in K. Show that every such element $a + b\alpha$ with $b \neq 0$ is the root of an irreducible quadratic polynomial in $F[x]$.
    (c) Show that every polynomial of degree 2 in $F[x]$ has a root in K.
    (d) Show that all the fields K constructed as above for a given prime p are isomorphic.
15. The polynomials $f(x) = x^3 + x + 1$, $g(x) = x^3 + x^2 + 1$ are irreducible over $\mathbb{F}_2$. Let K be the field extension obtained by adjoining a root of f, and let L be the extension obtained by adjoining a root of g. Describe explicitly an isomorphism from K to L.

*16. **(a)** Prove Lemma (6.21) for the case $F = \mathbb{C}$ by looking at the roots of the two polynomials.
    (b) Use the principle of permanence of identities to derive the conclusion when F is an arbitrary ring.

7. *Function Fields*

1. Determine a real polynomial in three variables whose locus of zeros is the projected Riemann surface (7.9).
2. Prove that the set $\mathcal{F}(U)$ of continuous functions on U' forms a ring.
3. Let $f(x)$ be a polynomial in $F[x]$, where F is a field. Prove that if there is a rational function $r(x)$ such that $r^2 = f$, then r is a polynomial.
4. Referring to the proof of Proposition (7.11), explain why the map $F \longrightarrow \mathcal{F}(S)$ defined by $g(x) \rightsquigarrow g(X)$ is a homomorphism.